ISRAEL JOURNAL OF MATHEMATICS 105 (1998), 337–348

LINEAR CORE CONDITIONS IN RESIDUALLY FINITE GROUPS

BY

L. LÉVAI*

Mathematical Institute, Hungarian Academy of Sciences
H-1364 Budapest, P.O.Box 127, Hungary
e-mail: levai@math-inst.hu

ABSTRACT

Let G be a residually finite or pro-finite group. We say that G satisfies the linear core condition with constant c if all finite index (open) subgroups of G contain a subgroup of index at most c which is normal in G. Answering a question of L. Pyber we give a complete characterisation of finitely generated residually finite and pro-finite groups satisfying a linear core condition. In the case of infinitely generated groups we prove that such groups are abelian-by-finite.

1. Introduction

In the past few years groups satisfying various core conditions have been much investigated. For example, by a recent result of Cutolo, Khukhro, Lennox, Rinauro, Smith and Wiegold [3], if in a locally finite group G the inequality $|H\colon \mathbf{Core}_G(H)| \leq c$ holds for all subgroups H of G and for some constant c (core-c groups in their terminology), then G has an abelian subgroup of c-bounded index (that is, its index is bounded from above by a function of c). For related results see [2], [10], [7], [9], [6]. Babai, Goodman and Pyber have considered finite and locally finite groups without core-free subgroups of order greater than c in [1]. It is proved there that in any finite group there are two

* Research supported by the Hungarian National Research Foundation (OTKA), grant no. 16432 and F023436.
Received December 4, 1996 and in revised form January 26, 1997

dedekind subgroups with the property that subgroups of order greater than $f(c)$ intersect nontrivially at least one of them. A dedekind subgroup in a group G is a subgroup D such that all subgroups of D are normal in G.

For residually finite and pro-finite groups it is more natural to consider only finite index (resp. open) subgroups. We say that a residually finite (resp. pro-finite) group G satisfies the linear core condition with constant c if for all finite index (resp. open) subgroups H of G we have $|H\colon \mathbf{Core}_G(H)| \leq c$, that is $|G\colon \mathbf{Core}_G(H)| \leq c|G\colon H|$. Note, that this definition is a special case of the f-core condition: Let G be a residually finite (resp. pro-finite) group and $f\colon \mathbf{N} \longrightarrow \mathbf{N}$ a function. We say that G satisfies the f-core condition if for all finite index (resp. open) subgroups H of G we have $|G\colon \mathbf{Core}_G(H)| \leq f(|G\colon H|)$. Residually finite groups satisfying a polynomial or exponential core condition or some related conditions have been studied by Lubotzky, Pyber and Shalev in [8]. It is noted there that for various residually finite groups G the intersection of all index n subgroups of G is of index at most n^c. This holds in particular for groups of finite upper rank, for $\mathrm{GF}(p)[[t]]$-analytic groups and for the Nottingham group.

László Pyber (personal communication) asked whether a group satisfying a linear core condition always has a dedekind subgroup of finite index. In the case of finitely generated groups we answer this question in the affirmative ($d(G)$, as usual, stands for the minimal number of generators of the group G, for the definition of linear core condition on cyclic subgroups see Definition 2.2):

THEOREM 3.1: *Let G be a finitely generated residually finite group. Then the following are equivalent:*

(i) *G satisfies a linear core condition;*

(ii) *G satisfies a linear core condition on cyclic subgroups;*

(iii) *G has an abelian dedekind subgroup of finite index.*

Moreover, if G satisfies the linear core condition on cyclic subgroups with constant c then G has an abelian dedekind subgroup of $c, d(G)$-bounded index.

The proof relies on the positive solution of the restricted Burnside problem by Zelmanov [11]. In fact it can be shown that the statement of Theorem 3.1 implies the positive solution of the restricted Burnside problem. Indeed, if G is a residually finite group of exponent c and G can be generated by d elements, then G satisfies the linear core condition on cyclic subgroups with constant c. Hence by Theorem 3.1, G has an abelian dedekind subgroup D of c, d-bounded index. D is an abelian group of exponent at most c and the number of generators of D is c, d-bounded thus the the order of D and then in turn the order of G is c, d-bounded, too.

We obtain various analogous results for pro-finite and pro-p groups. In particular we prove:

PROPOSITION 3.2: *Let p be an odd prime and G a finitely generated pro-p group. Then the following are equivalent:*
(i) *G satisfies a linear core condition;*
(ii) *G satisfies a linear core condition on cyclic subgroups;*
(iii) *the centre of G is open.*

In section 4 some simple examples are given which show that we cannot hope to find finite index dedekind subgroups in nonfinitely generated residually finite groups. However, using the result of [3] we can show

THEOREM 3.3: *There is a function $F\colon \mathbf{N} \longrightarrow \mathbf{N}$ such that if G is a pro-finite group satisfying the linear core condition with constant c then G has an open abelian normal subgroup A of index at most $F(c)$.*

As a corollary we have

COROLLARY 3.2: *Let G be a residually finite group satisfying the linear core condition with constant c. Then G has an abelian normal subgroup of index at most $F(c)$.*

2. Finite groups with a linear core condition

In this section we consider finite groups. First, however, we need two definitions:

Definition 2.1: Let G be any group (not necessarily finite). An element $g \in G$ is said to be a **normal element**, if $\langle g \rangle$ is a normal subgroup of G. A subgroup D of G is called a **dedekind subgroup** if every subgroup of D is normal in G. The fact that $D \leq G$ is a dedekind subgroup will be denoted by $D \,\mathit{ded}\, G$.

D being a dedekind subgroup is easily seen to be equivalent to saying that all elements of D are normal elements.

Definition 2.2: Let G be a group (not necessarily finite), and c a natural number. We say that G satisfies the linear core condition on cyclic subgroups with constant c if for all finite index normal subgroups N and for all $x \in G$ we have $|G\colon \mathbf{Core}_G(\langle x \rangle N)| \leq c|G\colon \langle x \rangle N|$.

To prove the main result of this section (Theorem 2.1) we need the following lemma:

LEMMA 2.1: *There exists a function $f_1\colon \mathbf{N}^2 \longrightarrow \mathbf{N}$ such that for any prime p and natural number k, if G is a finite p-group which is generated by normal elements $g_1, g_2, \ldots, g_n$ then there is a characteristic subgroup H in G such that $\exp(G/H)$ divides $f_1(k,p)$ and all elements of H are p^k-th powers in G.*

Proof: Let first p be odd. Clearly G/G^p is also generated by normal elements and has exponent p, so it is elementary abelian, therefore G is powerful. This implies the result with $H = G^{p^k}$ and $f_1(k,p) = p^k$. (See [4], Chapter 2 for properties of powerful p-groups.)

For $p = 2$ note that $G^m = \langle g_1^m, \ldots, g_n^m \rangle$ and $[g_i, g_j]$ commutes with both g_i and g_j. From this we get that $[g_i^t, g_j^s] = [g_i, g_j]^{ts}$ for all natural numbers s and t. On the other hand, if g_i and g_j do not commute modulo G^4 then they do not commute in $\langle g_i, g_j \rangle$ modulo $\langle g_i, g_j \rangle^4 = \langle g_i^4, g_j^4 \rangle$. Now this implies that $\langle g_i, g_j \rangle / \langle g_i^4, g_j^4 \rangle$ is a quaternion group of order 8 and so $g_i^2 \langle g_i^4, g_j^4 \rangle = g_j^2 \langle g_i^4, g_j^4 \rangle$. But then $g_i^{2+4s} g_j^{4t} = g_j^2$ for some natural numbers s and t, and from this we get $g_i^{2+4s} = g_j^{2-4t}$ showing $\langle g_i^2 \rangle = \langle g_j^2 \rangle$ in this case. Suppose now that for some $1 \leq i < j \leq n$ we would have $[g_i^2, g_j^2] \notin G^{16}$. Then $[g_i, g_j] \notin G^4$ and so $\langle g_i^2 \rangle = \langle g_j^2 \rangle$ forcing $[g_i^2, g_j^2] = 1$, a contradiction. This means that G^2 is powerful and as above we have that $f_1(k,2) = 2^{k+1}$ and all elements of $H = G^{2^{k+1}}$ are 2^k-th powers in G. ∎

THEOREM 2.1: *There exists a function $f_2\colon \mathbf{N} \longrightarrow \mathbf{N}$ such that for any natural number c, if G is a finite group satisfying the linear core condition on cyclic subgroups with constant c then G has an abelian dedekind subgroup D such that $\exp(G/D)$ divides $f_2(c)$.*

Proof: Let $g \in G$ be an arbitrary element. By the assumption on G we have a natural number $k \leq c$ with $\langle g^k \rangle \triangleleft G$. This means that $H = G^{c!}$ is a finite group generated by normal elements and hence it is nilpotent (it is a product of nilpotent normal subgroups). For each $p \in \pi(H)$ let us denote by S_p the Sylow p-subgroup of H. Note that if $p > c$, p odd, S_p is an abelian dedekind subgroup of G. Now let $p \leq c$. Let $k = \lfloor \log_2 c \rfloor + 1$; then $p^k > c$ and by Lemma 2.1 we have that there is a characteristic subgroup A_p of S_p such that all nontrivial elements of A_p are p^k-th powers (hence are normal elements) and $\exp(S_p/A_p)$ divides $f_1(k,p)$. Let $f(c) = c! \prod_{p \in \pi(H),\, p \leq c} f_1(k,p)$ and $D_0 = \prod_{p \in \pi(H)} A_p$ (for $p > c$ set $A_p = S_p$). Clearly, $\exp(G/D_0)$ divides $f(c)$. We will show that D_0 is a dedekind subgroup of G. Let $x \in D_0$ be an arbitrary element of D_0. Then $x = a_{p_1} a_{p_2} \cdots a_{p_t}$, $p_i \in \pi(D_0)$, $p_i \neq p_j$ for $i \neq j$ and $a_{p_i} \in A_{p_i}$. Now $\langle x \rangle = \prod_{i=1}^{t} \langle a_{p_i} \rangle$ and $\langle a_{p_i} \rangle \triangleleft G$. This means that all elements of D_0 are normal elements. As D_0 is Hamiltonian it contains an abelian subgroup of index at most 2. Thus $D = D_0^2$ is an abelian dedekind subgroup of G with $\exp(G/D)$ dividing $f_2(c) = 2f(c)$. ∎

COROLLARY 2.1: *There exists a function $f_3\colon \mathbf{N}^2 \longrightarrow \mathbf{N}$ such that if G is a finite group satisfying the linear core condition on cyclic subgroups with some constant c then G has an abelian dedekind subgroup A of index at most $f_3(d(G), c)$.*

Proof: Let G be a finite group satisfying the linear core condition on cyclic subgroups with some constant c. Then by Theorem 2.1 there is an abelian dedekind subgroup D of G with $\exp(G/D)$ dividing $f_2(c)$. By the positive solution of the restricted Burnside problem $|G\colon D| = |G/D| \leq B(d(G), f_2(c)) = f_3(d(G), c)$. ∎

Note that by a theorem of Cutolo, Khukhro, Lennox, Rinauro, Smith and Wiegold [3] a finite group satisfying the linear core condition with constant c has an abelian normal subgroup of c-bounded index. This provides a proof for Corollary 2.1 for finite groups satisfying the linear core condition (on all subgroups) which does not rely on the positive solution of the restricted Burnside problem. Note further that though the bound in Corollary 2.1 can be improved if G satisfies the linear core condition, even in this case no bound depending only on c exists as shown by Example 4.1.

3. Infinite groups with a linear core condition

In this section we consider infinite groups. Our first main result is

THEOREM 3.1: *Let G be a finitely generated residually finite group. Then the following are equivalent:*

(i) *G satisfies a linear core condition;*
(ii) *G satisfies a linear core condition on cyclic subgroups;*
(iii) *G has an abelian dedekind subgroup of finite index.*

Moreover, if G satisfies the linear core condition on cyclic subgroups with constant c then G has an abelian dedekind subgroup of $c, d(G)$-bounded index.

Proof: The implication (i)$\Longrightarrow$(ii) is trivial. We prove that (ii) implies (iii). Suppose that G satisfies the linear core condition on cyclic subgroups with constant c. Then each finite quotient of G is a finite $d(G)$-generator group satisfying the linear core condition on cyclic subgroups with constant c. Applying Corollary 2.1 we get for all finite index normal subgroups N of G an $A_N \triangleleft G$ with $N \leq A_N$ and $|G\colon A_N| \leq f_3(d(G), c)$, such that A_N/N is an abelian dedekind subgroup of

G/N. As the indices of the A_N's are bounded and G is finitely generated,

$$A = \bigcap_{\substack{N \lhd G \\ |G:N|<\infty}}$$

is of finite index in G; in fact its index is bounded by a function of c and $d(G)$. A is abelian because it is residually abelian, and is finitely generated as a finite index subgroup of a finitely generated group. Moreover, for any finite index normal subgroup N of G the quotient group AN/N is a dedekind subgroup of G/N. We claim that A is in fact a dedekind subgroup of G. To see this let B be any subgroup of A. A/B is a finitely generated abelian group so it follows from the fundamental theorem of abelian groups that A/B is residually finite. Hence

$$B = \bigcap_{\substack{B\le D\le A \\ |A:D|<\infty}} D = \bigcap_{\substack{M\le A \\ |A:M|<\infty}} BM = \bigcap_{\substack{N\le A \\ N\lhd G \\ |A:N|<\infty}} BN,$$

the last equality being valid since, if M is a finite index subgroup of A, then $\mathbf{Core}_G(M)$ is also a finite index subgroup of A. On the other hand, as A/N is a dedekind subgroup of G/N the quotient group BN/N is a normal subgroup of G/N, thus BN is a normal subgroup of G for all finite index normal subgroups N of G contained in A. Comparing this with the equality above we get that B itself is normal in G. This shows that A is a dedekind subgroup of G and, as we remarked earlier, its index is bounded in terms of c and $d(G)$.

Finally, suppose that G has a abelian dedekind subgroup D of finite index c. Now if M is a finite index subgroup of G then $|M\colon M\bigcap D| = |MD\colon D| \le |G\colon D| = c$ and $M\bigcap D \lhd G$, showing that G satisfies the linear core condition with constant c. ∎

We now turn to pro-finite groups. We use the standard notation $A \le_o B$ and $A \le_c B$ to abbreviate that A is an open or closed subgroup in B.

Definition 3.1: Let G be pro-finite group, and c a natural number. We say that G satisfies the linear core condition with constant c if for all open subgroups H of G the inequality $|G\colon \mathbf{Core}_G(H)| \le c|G\colon H|$ holds. Clearly, this is equivalent to saying that the core of any open subgroup of G has index at most c in the given subgroup. Moreover, we say that G satisfies the linear core condition on cyclic subgroups with constant c if for all open subgroups N and for all $x \in G$ we have $|G\colon \mathbf{Core}_G(\langle x\rangle N)| \le c|G\colon \langle x\rangle N|$.

Definition 3.2: Let G be a pro-finite group. An element $g \in G$ is said to be a **topologically normal element**, if $\overline{\langle g\rangle}$ is a normal subgroup of G. A closed subgroup D of G is called a **topological dedekind subgroup** if for any $H \leq_c D$ we have $H \lhd G$.

It is easy to see that D is a topological dedekind subgroup if and only if all elements of D are topologically normal elements. Note that in a pro-finite group a closed dedekind subgroup is always a topological dedekind subgroup; however, a topological dedekind subgroup may fail to be a dedekind subgroup.

THEOREM 3.2: *Let G be a pro-finite group. Then G satisfies a linear core condition on cyclic subgroups if and only if G has an abelian topological dedekind subgroup D with $\exp(G/D)$ finite. Moreover, if G satisfies a linear core condition on cyclic subgroups with constant c then $\exp(G/D)$ divides $f_2(c)$. (The function $f_2(c)$ was given in Theorem 2.1.)*

Proof: Suppose first that G satisfies the linear core condition on cyclic subgroups with constant c. Then each open quotient of G is a finite group satisfying the linear core condition on cyclic subgroups with constant c. Applying Theorem 2.1 we get for all open normal subgroups N of G an $A_N \lhd_o G$ with $N \leq A_N$, $\exp(G/A_N)$ dividing $f_2(c)$, A_N/N an abelian dedekind subgroup of G/N. Let

$$D = \bigcap_{N \lhd_o G} A_N;$$

D is abelian because it is residually abelian. We will prove first that D is a topological dedekind subgroup. Let $M \leq_c D$ and $x \in G$. For all open normal subgroups N of G we have $MN/N \operatorname{ded} G/N$, so $MN = M^x N$. Now

$$\overline{M} = \bigcap_{N \lhd_o G} MN = \bigcap_{N \lhd_o G} M^x N = \overline{M^x}.$$

This means that all $M \leq_c D$ are normal subgroups of G, proving $D \operatorname{ded} G$. It remains to prove that $\exp(G/D)$ divides $f_2(c)$. Let $x \in G$ be an arbitrary element. Then $x^{f_2(c)} \in A_N$ for all open normal subgroups N of G and hence $x^{f_2(c)} \in D$, as claimed.

Conversely, suppose that G has an open topological dedekind subgroup D with $\exp(G/D) = c$ finite. Let $x \in G$ be an arbitrary element. Then $x^c \in D$ and hence x^c is a topologically normal element. This implies that if N is an open normal subgroup of G then $\langle x^c \rangle N$ is an open normal subgroup too, and clearly $|\langle x \rangle N \colon \langle x^c \rangle N| \leq c$, showing that G satisfies the linear core condition on cyclic subgroups with constant c. ∎

A similar argument shows that for residually finite groups we have

PROPOSITION 3.1: *Let G be a residually finite group satisfying the linear core condition on cyclic subgroups with constant c. Then G has an abelian normal subgroup A such that $\exp(G/A)$ divides $f_2(c)$.*

It is not clear whether here A can be chosen to be a dedekind subgroup or not. The positive solution of the restricted Burnside problem implies

COROLLARY 3.1: *Let G be a finitely generated pro-finite group. Then the following are equivalent:*

(i) *G satisfies a linear core condition;*
(ii) *G satisfies a linear core condition on cyclic subgroups;*
(iii) *G has an open abelian topological dedekind subgroup.*

Proof: The first implication is trivial. The second implication follows from Theorem 3.2 and the positive solution of the restricted Burnside problem. To prove the implication (iii)$\Longrightarrow$(i) let D be an open topological dedekind subgroup of G and M any closed subgroup. Then $M \cap D$ is a closed normal subgroup and $|M\colon M\cap D| = |MD\colon D| \leq |G\colon D|$. ∎

For pro-p groups we can say even more:

PROPOSITION 3.2: *Let p be an odd prime and G a finitely generated pro-p group. Then the following are equivalent:*

(i) *G satisfies a linear core condition;*
(ii) *G satisfies a linear core condition on cyclic*
(iii) *the centre of G is open.*

Proof: The first implication is trivial. Suppose that G satisfies a linear core condition on cyclic subgroups. Then by Corollary 3.1, G contains an open abelian topological dedekind subgroup D. Let $g \in D$ be an element of infinite order and $C = \overline{\langle g\rangle} \cong \mathbf{Z}_p$. Then C is a closed normal subgroup of G on which G acts (via conjugation) as a finite group (because D is abelian). This defines a homomorphism from G into $\Gamma = \mathrm{Aut}(\mathbf{Z}_p)$, where the image is a finite p-group. We claim that Γ has no nontrivial finite p-subgroups. Indeed, let $\alpha \in \Gamma$ be an element of finite order p^t. Then α acts on C and its action is completely determined by the image of g, say $\alpha(g) = g^\mu$ for some $\mu \in \mathbf{Z}_p$. As $\alpha^{p^t}(g) = g$ we have $\mu^{p^t} = 1$, that is μ is a 1-unit in $\mathbf{Z}_p$. Now the group of 1-units is isomorphic (via the logarithm map) to the additive group of the maximal ideal of $\mathbf{Z}_p$, which is torsion free, forcing $p^t = 1$. This means that C is central in G. Let H be the closed subgroup of D generated by the elements of infinite order; then by the above argument H is central in G. We claim that H has finite index in G and hence it is open. As D has finite index in G it suffices to prove that H has finite index in D. Now this is true, because D/H is a finitely generated abelian torsion pro-p group.

Finally, the centre is clearly a topological dedekind subgroup, so by Corollary 3.1 G satisfies a linear core condition. ∎

Note that the groups in Example 4.2 show that neither Theorem 3.1 nor Corollary 3.1 holds for nonfinitely generated residually finite or pro-finite groups in general. However, we have

THEOREM 3.3: *There is a function $F\colon \mathbf{N} \longrightarrow \mathbf{N}$ such that if G is a pro-finite group satisfying the linear core condition with constant c then G has an open abelian normal subgroup A of index at most $F(c)$.*

Proof: Note first that by the Mal'cev Local Theorem (see, e.g., [5] Proposition 1.K.2) it is sufficient to prove that all finitely generated closed subgroups of G are abelian-by-bounded, so we may suppose that G is finitely generated.

The first step is a reduction to finitely generated central-by-finite pro-finite groups. First of all note that by Theorem 3.2 we have an abelian topological dedekind subgroup D in G such that $\exp(G/D)$ divides $f_2(c)$; thus, as remarked earlier, G/D is finite because G is finitely generated. Let π be the set of odd primes greater than $f_2(c)$. Then π' is finite and the Hall pro-π subgroup D_0 of D is the (unique normal) Hall pro-π subgroup of G. Let N_i ($i \in \mathbf{N}$) be a descending series of open normal subgroups of G forming a base of the neighbourhoods of the identity. Then D_0N_i/N_i is a normal Hall π-subgroup of G/N_i and, if X is a complement of D_0N_i/N_i in G/N_i, then X contains a normal subgroup of index at most c, thus $|G/N_i\colon C_{G/N_i}(D_0N_i/N_i)| \leq c$. Denoting by C_i the preimage of $C_{G/N_i}(D_0N_i/N_i)$ in G the C_i's form a descending chain of open normal subgroups and $|G\colon C_i| \leq c$ for all $i \in \mathbf{N}$. Hence $C = \bigcap_{i=0}^{\infty} C_i$ is an open normal subgroup of index at most c in G and clearly, $C = C_G(D_0)$. Thus we may suppose that $D_0 \leq Z(G)$. Now let $p \in \pi'$, let D_p be the unique Sylow pro-p subgroup of D and X_i the preimage of $C_{G/N_i}(D_pN_i/N_i)$. Then $X_i \geq D$ and G/X_i is a finite group of exponent dividing $f_2(c)$. Let $D_pN_i/N_i = \langle z_1, \ldots, z_n\rangle$, where the z_j's are independent. If $a \in G/N_i$ then there are $k_j \in \mathbf{N}$ such that $z_j^a = z_j^{k_j}$. Clearly, k_j is unique modulo $|z_j|$. Now $(z_iz_j)^a = (z_iz_j)^k = z_i^k z_j^k$ for some $k \in \mathbf{N}$, thus as $(z_iz_j)^a = z_i^a z_j^a = z_i^{k_i} z_j^{k_j}$ we have $k - k_i \equiv 0 \bmod |z_i|$ and $k - k_j \equiv 0 \bmod |z_j|$. Hence if z_l has maximal order among the z_j's, we can assign a natural number $k(a)$ to each $a \in G/X_i$ with $z_i^a = z_i^{k(a)}$ for $i = 1, 2, \ldots, n$ and thus $k(a)^{f_2(c)} \equiv 1 \bmod |z_l|$. Now the number of (incongruent) solutions to this congruence is at most $2f_2(c)$

($f_2(c)$ if p is odd). This implies that $|G/X_\imath| \le 2f_2(c)$, thus $|G\colon X_\imath|$ is bounded in terms of c and, as in the case of D_0, we conclude that $|G\colon C_G(D_p)|$ is bounded too. Since $|\pi'| \le f_2(c)$, $|G\colon C_G(D)|$ is bounded by a function of c, hence we may suppose that G is central-by-finite. In this case there are at most $|G\colon Z(G)|^2$ commutator elements in G. Hence we may choose an open normal subgroup N such that N does not contain nontrivial commutators. Then for any pair of elements x, $y \in G$, x and y commute modulo N if and only if they commute in G. By Theorem 1 of [3] the finite core-c group G/N has an abelian normal subgroup of c-bounded index. Thus if A denotes the preimage of this abelian normal subgroup, then A is an abelian normal subgroup of G and its index is bounded in terms of c. ∎

COROLLARY 3.2: *Let G be a residually finite group satisfying the linear core condition with constant c. Then G has an abelian normal subgroup of index at most $F(c)$.*

Proof: Let us denote by $\widehat{G}$ the pro-finite completion of G. Then $\widehat{G}$ satisfies the linear core condition with constant c. Thus by Theorem 3.3, $\widehat{G}$ has an open abelian normal subgroup A of index at most $F(c)$ and hence $G \cap A$ is an abelian normal subgroup of G of index at most $F(c)$. ∎

4. Some examples

Our first example shows that the bound in Corollary 2.1 must depend on $d(G)$.

Example 4.1: Let p be an odd prime number and define an n generator finite group G_n for all natural numbers n as follows. Let $A_n = \mathrm{C}_p^n$, $B_n = \mathrm{C}_{p^n}$, $M_n = A_n \times B_n$, the automorphism φ_n of M_n of order two be the identity on A_n and the inverse map on B_n and let $G_n = M_n\colon \langle\varphi_n\rangle$, the semidirect product. Then G_n satisfies the linear core condition with constant $2p$, but any dedekind subgroup D of G_n has index at least p^n.

Proof: Let us fix a natural number n. (We will omit the subscript n.) Let $H \le G$; we must show that $|H\colon \mathbf{Core}_G(H)| \le 2p$. Clearly, $|H\colon H \cap M| \le 2$, so it suffices to prove that if $K \le M$ then $|K\colon \mathbf{Core}_G(K)| \le p$. $K \cap B \triangleleft G$; we will denote the image of an arbitrary subgroup X of G under the natural projection of G onto $G/(K \cap B)$ by $\overline{X}$. As $\overline{K} \cap \overline{B} = 1$ and $\overline{M}^p \le \overline{B}$ we have $\overline{K}^p = 1$, that is $\overline{K} \le \mathrm{Soc}(\overline{M})$. Now $\mathrm{Soc}(\overline{M})$ is a vector space over the p element field, so $\dim(\overline{K} \cap \overline{A}) + \dim(\overline{KA}) = \dim(\overline{K}) + \dim(\overline{A})$. Hence $\dim(\overline{K} \cap \overline{A}) \ge \dim(\overline{K}) - 1$.

This means that $|\overline{K}\colon (\overline{K} \cap \overline{A})| \leq p$. Let us denote by N the preimage of $\overline{K} \cap \overline{A}$ in G. Then $N \leq K$, $|K\colon N| \leq p$ and $N \triangleleft G$.

It remains to prove that any dedekind subgroup D of G has index at least p^n. Let D be a dedekind subgroup of G. We claim that $M \cap D \leq A$ or $M \cap D \leq B$. Indeed, let us suppose that there is an $x = ab \in D$ with $a \in A$, $b \in B$, $a \neq 1 \neq b$. Then $x^\varphi = x^m$ for some $m \in \mathbf{Z}$. Now $x^\varphi = a^\varphi b^\varphi = ab^{-1}$ and $x^m = a^m b^m$. From this we get $m \equiv 1 \bmod p$ and $m \equiv -1 \bmod |b|$, hence p divides $(m+1) - (m-1)$, that is p divides 2, a contradiction. ∎

The second example shows that in Theorem 3.1 and in Corollary 3.1 it is essential to assume that G is finitely generated, thus the answer to the question of László Pyber is negative in the case of infinitely generated groups.

Example 4.2: Let p be an odd prime and let $A = \mathrm{C}_p^{\aleph_0}$, $B = \mathbf{Z}_p$, φ the automorphism of $A \times B$ which is the identity on A and the inverse map on B, and $G \cong (A \times B)\colon \langle\varphi\rangle$. Then G is a residually finite group satisfying the linear core condition with constant $2p$ but G has no open topogical dedekind subgroup, and no finite index dedekind subgroup.

Proof: Indeed, the groups G_n in Example 4.1 form an inverse system of finite groups and G is the inverse limit of this system. As a residually finite group G satisfies the linear core condition with constant $2p$, because if $X \leq G$ is a finite index subgroup of G, then $X \cap A$ and $X \cap B$ are normal subgroups of G and $\overline{G} = G/((X \cap A)(X \cap B))$ is a finite group which is a subgroup of G_n for some n. G cannot have an open topological dedekind subgroup D of finite index c, because the image of D would be a dedekind subgroup of index at most c in G_n for all n. On the other hand, G cannot have a dedekind subgroup of finite index, because if D is any dedekind subgroup and $M = A \times B$ then $D \cap M$ is contained in either A or B. Indeed, if $x = ab \in D \cap M$ then $x^\varphi = (ab)^\varphi = ab^{-1}$ and, as D is a dedekind subgroup, $x^\varphi = x^m$ for some integer m. That is $(ab)^m = ab^{-1}$ and, from this, $a^{m-1} = b^{-1-m}$. This can be true only if $b = 1_G$ or $m + 1 = 0$ and $a^2 = 1_G$. The latter implies $a = 1_G$ (note that p was odd). ∎

Acknowledgement: The author is deeply indebted to László Pyber and Péter Pál Pálfy for their encouragement and contribution to this paper in the form of useful suggestions, remarks and questions. The thorough work of the referee and his suggestion for simplifying the proof of Theorem 3.1 is also gratefully acknowledged.

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